

FIELDS OF MODULI AND FIELDS OF DEFINITION OF ODD SIGNATURE CURVES

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ABSTRACT. Let X be a smooth projective curve of genus $g \geq 2$ defined over a field K . We show that X can be defined over its field of moduli K_X if the signature of the covering $X \rightarrow X/\text{Aut}(X)$ is of type $(0; c_1, \dots, c_k)$, where some c_i appears an odd number of times. This result is applied to q -gonal curves and to plane quartics.

INTRODUCTION

Let X be a smooth projective curve of genus g defined over a field K and let K_X be its field of moduli (see Section 1, Definition 1.1). It is well known that X can be defined over K_X if either $g = 0, 1$ or the automorphism group of X is trivial. However, there are examples of curves which can not be defined over K_X , as first observed by Earle and Shimura in [5, 14]. In [10] B. Huggins studied this problem for hyperelliptic curves in characteristic $p \neq 2$, proving that a hyperelliptic curve X of genus $g \geq 2$ with hyperelliptic involution ι can be defined over K_X provided that $\text{Aut}(X)/\langle \iota \rangle$ is not cyclic or is cyclic of order divisible by p .

The first examples of non-hyperelliptic curves not definable over their field of moduli have been given in [10] and [7].

Recently R. Hidalgo [6] considered complex curves X such that the natural covering $\pi_X : X \rightarrow X/\text{Aut}(X)$ has signature of the form $(0; a, b, c, d)$, proving that X can be defined over its field of moduli if $d \notin \{a, b, c\}$. In this paper we observe that such result can be extended to *odd signature curves*, i.e. curves such that the signature of π_X is of the form $(0; c_i, \dots, c_r)$ where some c_i appears exactly an odd number of times. More precisely, we prove the following result, which is a consequence of [4, Theorem 3.1].

Theorem 0.1. *Let X be a smooth projective curve of genus $g \geq 2$ defined over a field K . If X is an odd signature curve, then K_X is a field of definition for X .*

This result implies that non-normal q -gonal curves can be defined over their field of moduli and that plane quartics can be defined over their field of moduli if $|\text{Aut}(X)| > 4$. In the last section of the paper we construct examples of plane quartics with $\text{Aut}(X) \cong C_2$ which can not be defined over their field of moduli and we prove that, in case $\text{Aut}(X) \cong C_2 \times C_2$, the field of moduli relative to the extension \mathbb{C}/\mathbb{R} is always a field of definition. This implies the following.

Theorem 0.2. *Let X be a smooth plane quartic over \mathbb{C} which is isomorphic to its conjugate. If $\text{Aut}(X)$ is not cyclic of order two, then X can be defined over \mathbb{R} .*

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1. PRELIMINARIES

Let X be a smooth projective curve defined over a field K . A subfield N of K is a *field of definition* of X if there exists a curve X' defined over N such that X' is isomorphic to X over K . Moreover, we say that X is *definable* over N if there exists a curve X' defined over N such that X' is isomorphic to X over \bar{K} .

Definition 1.1. Let K be a field, \bar{K} be an algebraic closure of K and X be a curve defined over K . The *field of moduli* K_X of X is the intersection of all fields of definition of X , seen as a curve over \bar{K} .

Another definition for the field of moduli, relative to a given field extension F/L , is given as follows. If $P \in F[x_0, \dots, x_n]$ and $\sigma \in \text{Aut}(F/L)$, then P^σ denotes the polynomial obtained by applying σ to the coefficients of P . If a curve X is defined as the zero locus of the homogeneous polynomials $P_1, \dots, P_s \in F[x_0, \dots, x_n]$, then the polynomials $P_1^\sigma, \dots, P_s^\sigma$ define a new smooth projective curve X^σ .

Definition 1.2. The *field of moduli* of X relative to the extension F/L , denoted by $M_{F/L}(X)$, is the fixed field of the group

$$U_{F/L}(X) := \{\sigma \in \text{Aut}(F/L) : X \text{ is isomorphic to } X^\sigma \text{ over } F\}.$$

Let P be the prime field of K . By a theorem of Koizumi (see [11] and [9, Theorem 1.5.8]) the field of moduli $M_{\bar{K}/P}(X)$ is a purely inseparable extension of the field of moduli K_X . In particular these two fields coincide if K is a perfect field. For example, if $K = \mathbb{C}$, then $K_X = M_{\mathbb{C}/\mathbb{Q}}(X)$. The relationship between K_X and the fields of moduli of X relative to Galois extensions is given by the following result (see [9, Theorem 1.6.8]).

Theorem 1.3. *Let X be a smooth projective algebraic curve defined over a field K and K_X be the field of moduli of X . Then X is definable over K_X if and only if given any algebraically closed field $F \supseteq K$, and any subfield $L \subseteq F$ with F/L Galois, X (seen as a curve over F) can be defined over the field $M_{F/L}(X)$.*

Given a smooth projective algebraic curve Y defined over L , a branched covering $\phi : X \rightarrow Y$ defined over F and $\sigma \in \text{Aut}(F/L)$, we denote by $\phi^\sigma : X^\sigma \rightarrow Y^\sigma$ the branched covering obtained by applying σ to the defining polynomials of ϕ .

Assume now that a curve L is a field of definition of a curve X over F , i.e. there exists an isomorphism $g : X \rightarrow Y$, where Y is a curve defined over L . If $\sigma \in \text{Aut}(F/L)$, then $f_\sigma := (g^\sigma)^{-1} \circ g : X \rightarrow X^\sigma$ is an isomorphism (observe that $Y = Y^\sigma$) and $f_{\tau\sigma} = f_\sigma^\tau \circ f_\tau$ holds for all $\sigma, \tau \in \text{Aut}(F/L)$. The following theorem by A. Weil shows that the latter condition is also sufficient for the field L to be a field of definition for X .

Theorem 1.4 (Weil [15]). *Let X be a smooth projective algebraic curve defined over a field F and let F/L be a Galois extension. If for every $\sigma \in \text{Aut}(F/L)$ there is an isomorphism $f_\sigma : X \rightarrow X^\sigma$ defined over F such that the compatibility condition $f_{\tau\sigma} = f_\sigma^\tau \circ f_\tau$ holds for all $\sigma, \tau \in \text{Aut}(F/L)$, then there exist a smooth projective algebraic curve Y defined over L and an isomorphism $g : X \rightarrow Y$ defined over F such that $g^\sigma \circ f_\sigma = g$.*

The following result by Dèbes-Emsalem is a consequence of Weil's theorem and provides a sufficient condition for the curve X to be defined over the field $M_{F/L}(X)$ (see [3, §2.4] for the definition of field of moduli of a covering).

Theorem 1.5 (Dèbes-Emsalem [4]). *Let F/L be a Galois extension and X be a smooth projective curve of genus $g \geq 2$ defined over F with $L := M_{F/L}(X)$. Then there exist a smooth projective curve B defined over L and a Galois branched covering $\phi : X \rightarrow B$ defined over F , with $\text{Aut}(X)$ as its deck group, so that $M_{F/L}(\phi) = L$. Moreover, if B contains at least one L -rational point outside of the branch locus of ϕ , then L is also a field of definition of X .*

Remark 1.6. The condition $L := M_{F/L}(X)$ in Theorem 1.5 is not restrictive since by [4, Proposition 2.1] the field of moduli relative to the extension $F/M_{F/L}(X)$ is $M_{F/L}(X)$.

2. PROOF OF THE THEOREM

Let $\phi : X \rightarrow X/G$ be a branched Galois covering between smooth projective curves and let q_1, \dots, q_r be its branch points. The *signature* of ϕ is defined as $(g_0; c_1, \dots, c_r)$, where g_0 is the genus of X/G and c_i is the ramification index of any point in $\phi^{-1}(q_i)$. The *branch divisor* of ϕ , denoted by $D(\phi)$, is the divisor of X/G defined by $D(\phi) = \sum_{i=1}^r c_i q_i$.

Definition 2.1. A smooth projective curve X of genus $g \geq 2$ has *odd signature* if the signature of the covering $\pi_X : X \rightarrow X/\text{Aut}(X)$ is of the form $(0; c_1, \dots, c_r)$ where some c_i appears exactly an odd number of times.

Definition 2.2. Let B be a smooth projective curve defined over a field L . A divisor $D = p_1 + \dots + p_r$ of B is called *L -rational* if for each $\sigma \in \text{Aut}(\bar{L}/L)$ we have that $D^\sigma := \sigma(p_1) + \dots + \sigma(p_r) = D$.

The following is an easy consequence of Riemann-Roch theorem and the fact that a curve of genus zero with a L -rational point is isomorphic to $\mathbb{P}^1(L)$ (see also [9, Lemma 4.0.4.]).

Lemma 2.3. *Let B be a smooth projective curve of genus 0 defined over an infinite field L and suppose that B has an L -rational divisor D of odd degree. Then B has infinitely many L -rational points.*

Lemma 2.4. *Given a Galois branched covering $\phi : X \rightarrow X/G$ as before defined over F , we have $D(\phi^\sigma) = D(\phi)^\sigma$ for any $\sigma \in \text{Aut}(F/L)$.*

Proof. Observe that $\sigma \circ \phi = \phi^\sigma \circ \sigma$, where we denote by σ the bijection acting as σ on the coordinates of the points of X and X/G . Thus q_i belongs to the support of $D(\phi)$ if and only if $\sigma(q_i)$ is in the support of $D(\phi^\sigma)$ and the fibers over the two points have the same cardinality. \square

The proof of Theorem 0.1 follows from Theorem 1.3 and the following result.

Theorem 2.5. *Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field F and let $L \subset F$ be a subfield such that F/L is Galois. If X is an odd signature curve, then $M_{F/L}(X)$ is a field of definition for X .*

Proof. By Remark 1.6 we can assume that $M_{F/L}(X) = L$. By Theorem 1.5 there exists a canonical L -model B of $X/\text{Aut}(X)$ and a commutative diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{f_\sigma} & X^\sigma \\
 \pi_X \downarrow & & \downarrow \pi_X^\sigma \\
 X/\text{Aut}(X) & \xrightarrow{h_\sigma} & (X/\text{Aut}(X))^\sigma \\
 & \searrow g & \swarrow g^\sigma \\
 & B &
 \end{array}$$

where $\sigma \in \text{Aut}(F/L)$ (this coincides with $U_{F/L}$ by [4, Proposition 2.1]) and f_σ, h_σ, g are isomorphisms. Let $\phi = g \circ \pi_X$. The fact that f_σ is an isomorphism and Lemma 2.4 imply that $D(\phi) = D(\phi^\sigma) = D(\phi)^\sigma$, i.e. $D(\phi)$ is an L -rational divisor. Also, as g is an isomorphism, $D(\phi) = g(D(\pi_X))$ and ϕ has the same signature of π_X . If q_1, \dots, q_{2k+1} are the points in the support of $D(\phi)$ with the same coefficient c_i , then the divisor $q_1 + \dots + q_{2k+1}$ is an L -rational divisor of odd degree.

If L is infinite this implies, by Lemma 2.3, that B has an L -rational point outside of the branch locus of ϕ , thus X can be defined over L by Theorem 1.5. In case L is finite the result follows from [10, Corollary 2.11]. \square

3. CYCLIC q -GONAL CURVES

Let F be an algebraically closed field of characteristic $p \neq 2$ and let X be an algebraic curve of genus $g \geq 2$ defined over F . If the automorphism group of X contains a cyclic subgroup C_q , where q is a prime number, such that X/C_q has genus zero, then the curve is called a *cyclic q -gonal curve*. If in addition C_q is normal in $\text{Aut}(X)$, then X is called a *normal cyclic q -gonal curve*. In this case the *reduced automorphism group* $\overline{\text{Aut}(X)} := \text{Aut}(X)/C_q$ is isomorphic to a finite subgroup of $\text{PGL}_2(F)$.

In case $\overline{\text{Aut}(X)}$ is not cyclic B. Huggins [10, Theorem 5.3] and A. Kontogeorgis [12, Proposition 3.2] proved the following theorem.

Theorem 3.1. *Let K be a perfect field of characteristic $p \neq 2$ and let F be an algebraic closure of K . Let X be a normal cyclic q -gonal curve over F such that $\overline{\text{Aut}(X)}$ is not cyclic or that $\overline{\text{Aut}(X)}$ is cyclic of order divisible by p . Then X can be defined over its field of moduli relative to the extension F/K .*

In case $\overline{\text{Aut}(X)}$ is cyclic of order n and $p = 0$, then X is isomorphic to a curve with equation $y^q = f(x)$, where f is as given in Table 3. Observe that $\overline{\text{Aut}(X)}$ is generated by $\nu(x) = \zeta_n x$, where ζ_n is a primitive n -th root of unity. The three cases in Table 3 differ by the number N of branch points of the cover $X \rightarrow X/C_q$ fixed by ν .

Corollary 3.2. *Let X be a normal cyclic q -gonal curve of genus $g \geq 2$ defined over a field K of characteristic zero such that $\overline{\text{Aut}(X)}$ is cyclic of order $n \geq 2$ and let N be as above. If either $N = 1$, or $N = 0$ and $\frac{2g-2+2q}{n(q-1)}$ is odd, or $N = 2$ and $\frac{2g}{n(q-1)}$ is odd, then X is definable over K_X .*

Proof. The signature of the covering $\pi_X : X \rightarrow X/\text{Aut}(X)$ is given in Table 1. If $N = 1$ then clearly X has odd signature. Otherwise, if $N = 0$, the number of branch points with ramification index q equals $\frac{2g-2+2q}{n(q-1)}$ by the Riemann-Hurwitz

formula, thus again X has odd signature. Similarly for $N = 2$. Thus the result follows from Theorem 0.1. \square

N	signature of π_X	$f(x)$
0	$(0; n, n, q, \dots, q)$	$x^{nt} + \dots + a_i x^{n(t-i)} + \dots + a_{t-1} x^n + 1$ where $q nt$
1	$(0; n, nq, q, \dots, q)$	$x^{nt} + \dots + a_i x^{n(t-i)} + \dots + a_{t-1} x^n + 1$ where $q \nmid nt$
2	$(0; nq, nq, q, \dots, q)$	$x(x^{nt} + \dots + a_i x^{n(t-i)} + \dots + a_{t-1} x^n + 1)$ where $q \nmid nt + 1$

TABLE 1. Cyclic q -gonal curves with $\overline{\text{Aut}(X)} = C_n$

We will now construct examples of cyclic q -gonal curves not definable over their field of moduli following [9, 10]. Let $m, n > 1$ be two integers, $a_1, \dots, a_m \in \mathbb{C}$ and consider the polynomial

$$(1) \quad f(x) := \prod_{1 \leq i \leq m} (x^n - a_i)(x^n + 1/\bar{a}_i).$$

We will look for such an f with the following properties: $|a_i| \neq |a_j|$ if $i \neq j$, $a_i/\bar{a}_i \neq a_j/\bar{a}_j$ if $i \neq j$, $|a_i| \neq |1/a_j|$ for all i, j , $f(0) = -1$. Moreover, if $n = 3$ we ask that the following automorphism does not map the zero set of f into itself:

$$\tau : x \mapsto \frac{-(x - \sqrt{3} - 1)}{x(\sqrt{3} - 1) + 1}.$$

We observe that such polynomials exist for any m, n : for $n \neq 3$ we can consider

$$f(x) = \prod_{1 \leq l \leq m} (x^n - (l+1)\kappa^l)(x^n + \frac{\kappa^l}{l+1}),$$

and for $n = 3$ the polynomial:

$$f(x) = (x^3 - \alpha^3)(x^3 + \frac{1}{\alpha^3}) \prod_{1 \leq l \leq m-1} (x^3 - (l+1)\kappa^l)(x^3 + \frac{\kappa^l}{l+1}),$$

where κ is a primitive m -th root of $(-1)^{m-1}$ and $\alpha = -(2 + \sqrt{3})$ (observe that $\tau(\alpha) = \alpha$).

Lemma 3.3. *Let X be a cyclic q -gonal curve over \mathbb{C} given by $y^q = f(x)$, where f is as in (1) and satisfies the properties mentioned above. Then:*

- i) $\text{Aut}(X)$ is generated by $\iota(x, y) = (x, \zeta_q y)$ and $\nu(x, y) = (\zeta_n x, y)$;
- ii) the signature of π_X is $(0; q, \dots, q, n, n)$ if $q|2mn$ and $(0; q, \dots, q, n, qn)$ otherwise, where q appears $2m$ -times.

Proof. Observe that ii) is obvious by Table 1. If $n \neq 3$, then i) follows from [10, Lemma 6.1] and its proof (which does not depend on the fact that m is odd). For $n = 3$ we need to exclude the missing case $\langle \bar{\nu} \rangle < \overline{\text{Aut}(X)} \cong A_4$, where $\bar{\nu}$ is the image of ν in $\overline{\text{Aut}(X)}$. Suppose we are in this case, then by [2, Corollary 3.2] τ would be an automorphism of $f(x)$, giving a contradiction. \square

q	signature of π_X	g	$\text{Aut}(X)$
3	(0; 2, 3, 8)	2	$\text{GL}(2, 3)$
3	(0; 2, 3, 12)	3	$\text{SL}(2, 3)/\text{CD}$
5	(0; 2, 4, 5)	4	S_5
7	(0; 2, 3, 7)	3	$\text{PSL}(2, 7)$
$q \geq 5$	(0; 2, 3, $2q$)	$\frac{(q-1)(q-2)}{2}$	$(C_q \times C_q) \rtimes S_3$
$q \geq 3$	(0; 2, 2, 2, q)	$(q-1)^2$	$(C_q \times C_q) \rtimes V_4$
$q \geq 3$	(0; 2, 4, $2q$)	$(q-1)^2$	$(C_q \times C_q) \rtimes D_4$

TABLE 2. Non-normal q -gonal curves.

The following generalizes [9, Proposition 5.0.5] and [10, Proposition 6.2]. Observe that if q does not divide mn , then X is an odd signature curve by the previous Lemma, thus it can be defined over its field of moduli relative to the extension \mathbb{C}/\mathbb{R} .

Proposition 3.4. *Let X be a cyclic q -gonal curve over \mathbb{C} given by $y^q = f(x)$, where $q > 2$, f is as in (1) and satisfies the properties mentioned above, $m, n > 1$ and $q|mn$. The field of moduli of X relative to the extension \mathbb{C}/\mathbb{R} is \mathbb{R} and is a field of definition of X if and only if n is odd.*

Proof. Observe that X is isomorphic to the conjugate curve

$$\bar{X} : y^q = \prod_{1 \leq i \leq m} (x^n - \bar{a}_i)(x^n + 1/a_i)$$

by the isomorphism

$$\mu(x, y) = \left(\frac{1}{\zeta_{2n}x}, \frac{\zeta_{2q}y}{x^{2mn/q}} \right).$$

By Lemma 3.3 the automorphism group of X is generated by ι and ν , thus any isomorphism between X and \bar{X} is of the form $\mu^j \nu^k$, where $0 \leq j \leq q-1$ and $0 \leq k \leq n-1$. An easy computation shows that

$$\overline{(\mu \nu^k)} \mu \nu^k = (\tau')^{2k+1} \nu^{2k+1},$$

where $\tau'(x, y) = (x, \zeta_n^{mn/q} y)$. Moreover, since ι commutes with μ and ν :

$$\overline{(\mu^j \nu^k)} \mu^j \nu^k = \bar{\mu} \iota^{-j} \bar{\nu}^k \mu^j \nu^k = \bar{\mu} \nu^k \mu \nu^k = \overline{(\mu \nu^k)} \mu \nu^k.$$

In case n is even the cocycle condition in Theorem 1.4 does not hold since $\nu^{2k+1} \neq id$ for any k , thus X cannot be defined over \mathbb{R} . Otherwise, if n is odd, we have $\overline{(\mu \nu^k)} \mu \nu^k = id$ with $k = (n-1)/2$, so that X can be defined over \mathbb{R} . \square

Corollary 3.5. *Let X be a non-normal q -gonal curve defined over a field K of characteristic zero. Then X is definable over K_X .*

Proof. By [16, Theorem 8.1] the signature of π_X is given in Table 2. In any case X has odd signature, thus the result follows from Theorem 0.1. \square

4. PLANE QUARTICS

In this section X will be a smooth plane quartic defined over an algebraically closed field of characteristic zero. Table 3 lists all possible automorphism groups of smooth plane quartics. Moreover, for each group, it gives the equation of a plane

quartic having this group as automorphism group (n.a. means “not above”, i.e. not isomorphic to other models above it in the table) and the signature of the covering π_X (see [1, Theorem 16 and §2.3]).

G	equation	signature
$\mathrm{PSL}_2(7)$	$z^3y + y^3x + x^3z$	$(0; 2, 3, 7)$
S_3	$z^4 + az^2yx + z(y^3 + x^3) + by^2x^2$ $a \neq b, ab \neq 0$	$(0; 2, 2, 2, 2, 3)$
$C_2 \times C_2$	$x^4 + y^4 + z^4 + ax^2y^2 + bx^2z^2 + cy^2z^2$ $a \neq b, a \neq c, b \neq c$	$(0; 2, 2, 2, 2, 2)$
D_4	$x^4 + y^4 + z^4 + az^2(y^2 + x^2) + by^2x^2$ $a \neq b, a \neq 0$	$(0; 2, 2, 2, 2, 2)$
S_4	$x^4 + y^4 + z^4 + a(z^2y^2 + z^2x^2 + y^2x^2)$ $a \neq 0, \frac{-1 \pm \sqrt{-7}}{2}$	$(0; 2, 2, 2, 3)$
$C_4^2 \rtimes S_3$	$z^4 + y^4 + x^4$	$(0; 2, 3, 8)$
$C_4 \odot (C_2)^2$	$z^4 + y^4 + x^4 + az^2y^2$ $a \neq 0, \pm 2, \pm 6, \pm(2\sqrt{-3})$	$(0; 2, 2, 2, 4)$
$C_4 \odot A_4$	$x^4 + y^4 + xz^3$	$(0; 2, 3, 12)$
C_6	$z^4 + az^2y^2 + y^4 + yx^3$ $a \neq 0$	$(0; 2, 3, 3, 6)$
C_9	$z^4 + zy^3 + yx^3$	$(0; 3, 9, 9)$
C_3	$z^3L_1(y, x) + L_4(y, x)$ (n.a.)	$(0; 3, 3, 3, 3, 3)$
C_2	$z^4 + z^2L_2(y, x) + L_4(y, x)$ (n.a.)	$(1; 2, 2, 2, 2)$

TABLE 3. Automorphisms of plane quartics.

Table 3 and Theorem 2.5 imply the following result.

Corollary 4.1. *Let X be a smooth plane quartic defined over an algebraically closed field K of characteristic zero. If either $\mathrm{Aut}(X)$ is trivial or $|\mathrm{Aut}(X)| > 4$, then X is definable over K_X .*

Observe that the hypothesis in the Corollary is equivalent to ask that $\mathrm{Aut}(X)$ is not isomorphic to either C_2 or $C_2 \times C_2$. We will now construct a plane quartic X with $\mathrm{Aut}(X) \cong C_2$ and of field of moduli \mathbb{R} but not definable over \mathbb{R} . Consider the family X_{a_1, a_2, a_3} of plane quartics defined by

$$y^4 + y^2(x - a_1z)(x + \frac{1}{a_1}z) + (x - a_2z)(x + \frac{1}{a_2}z)(x - a_3z)(x + \frac{1}{a_3}z) = 0,$$

where $a_1 \in \mathbb{R}$ and $a_2a_3 \in \mathbb{R}$. The following Lemma implies that the generic curve in the family is smooth and has automorphism group of order two.

Lemma 4.2. *The plane quartic X_{a_1, a_2, a_3} with $a_1 = 1, a_2 = 1 - i$ and $a_3 = 2(i - 1)$ is smooth and its automorphism group is generated by $\nu(x : y : z) = (x : -y : z)$.*

Proof. We recall that any automorphism of a smooth plane quartic is induced by an element of $\mathrm{PGL}(3, \mathbb{C})$. If $\mathrm{Aut}(X)$ properly contains the cyclic group generated by ν , then it contains a subgroup isomorphic to either $C_2 \times C_2, C_6$ or S_3 by [1, pag. 26]. We will now exclude each of these cases.

The first case can be excluded because an explicit computation shows that there is no involution, except ν , which preserves the four fixed points of ν .

Now suppose that $\text{Aut}(X)$ contains a cyclic subgroup of order 6 generated by α with $\nu = \alpha^3$. The automorphism $\tau := \alpha^2$ induces an order three automorphism $\bar{\tau}$ on the elliptic curve $E := X/\langle \nu \rangle$ having fixed points. This is a contradiction since the curve E (whose equation can be obtained replacing y^2 with y in the equation of X) has j -invariant distinct from zero.

Finally, suppose that $\text{Aut}(X)$ contains a subgroup $\langle \nu, \gamma \rangle$ isomorphic to S_3 . Here we will apply a method suggested by F. Bars [1]. By [1, Theorem 29], up to a change of coordinates the equation of X takes the following form:

$$(u^3 + v^3)w + u^2v^2 + auvw^2 + bw^4 = 0.$$

and the generators of S_3 with respect to the coordinates (u, v, w) are

$$\alpha := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta := \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus there exists $A \in \text{PGL}(3, \mathbb{C})$ such that $A\alpha A^{-1} = \nu, A\beta A^{-1} = \gamma$. The first condition implies that A is an invertible matrix of the following form

$$A = \begin{pmatrix} a & a & c \\ d & -d & 0 \\ g & g & l \end{pmatrix}.$$

Note that X has exactly four bitangents $x = s_j z$, $j = 1, 2, 3, 4$ invariant under the action of the involution ν , where s_j are the zeros of

$$\Delta = (x^2 - 1)^2 - 4(x - (1 + i))(x + \frac{1}{1 - i})(x - 2(-1 + i))(x - \frac{1}{2(1 + i)}).$$

Let $b_{j1} = (s_j, q_j, 1), b_{j2} = (s_j, -q_j, 1)$ be the two tangency points of the line $x = s_j z$. On the other hand, observe that the line $w = 0$ is invariant for α and it is bitangent to X at $p_1 = (1 : 0 : 0), p_2 = (0 : 1 : 0)$. Thus for some j we have $\{Ap_1, Ap_2\} = \{b_{j1}, b_{j2}\}$, from which we get $a = s_j g, d = \pm q_j g$. By means of these remarks and using the Magma [13] code available at this webpage

<https://sites.google.com/site/squispeme/home/fieldsmoduli>

we proved that $\gamma = A\beta A^{-1}$ is not an automorphism of X . \square

Proposition 4.3. *Let X_{a_1, a_2, a_3} as defined previously with $\text{Aut}(X_{a_1, a_2, a_3}) \cong C_2$. Then the field of moduli of X_{a_1, a_2, a_3} relative to the extension \mathbb{C}/\mathbb{R} is \mathbb{R} and is not a field of definition for X .*

Proof. Observe that the following is an isomorphism between $X := X_{a_1, a_2, a_3}$ and its conjugate \bar{X} :

$$\mu(x : y : z) = (-z : iy : x).$$

Since $\text{Aut}(X)$ is generated by $\nu(x : y : z) = (x : -y : z)$, the only isomorphisms between X and \bar{X} are μ and $\mu\nu$. Observe that $\bar{\mu}\mu = \nu$ and $\overline{(\mu\nu)}\mu\nu = \nu$. Therefore Weil's cocycle condition from Theorem 1.4 does not hold, so X cannot be defined over \mathbb{R} . \square

Finally we study plane quartics with automorphism group isomorphic to $C_2 \times C_2$, which belong to the following family:

$$X_{a,b,c} : x^4 + y^4 + z^4 + ax^2y^2 + bx^2z^2 + cy^2z^2 = 0,$$

where $a, b, c \in \mathbb{C}$. It can be easily checked that $X_{a,b,c}$ is smooth unless $a^2 + b^2 + c^2 - abc = 4$ or some of a^2, b^2, c^2 is equal to 4. A subgroup of $\text{Aut}(X_{a,b,c})$ isomorphic to $C_2 \times C_2$ is generated by the involutions:

$$\iota_1(x : y : z) = (-x : y : z), \quad \iota_2(x : y : z) = (x : -y : z).$$

We will denote by $G \cong S_3 \ltimes (C_2 \times C_2)$ the group acting on the triples $(a, b, c) \in \mathbb{C}^3$ generated by

$$\begin{aligned} g_1(a, b, c) &= (b, a, c), & g_2(a, b, c) &= (b, c, a), \\ g_3(a, b, c) &= (-a, -b, c), & g_4(a, b, c) &= (a, -b, -c). \end{aligned}$$

The following comes from a result by E.W. Howe [8, Proposition 2], observing that any isomorphism between $X_{a,b,c}$ and $X_{g(a,b,c)}$, $g \in G$, is defined over $\mathbb{Q}(i)$.

Proposition 4.4. *If a^2, b^2, c^2 are pairwise distinct, then $\text{Aut}(X_{a,b,c}) \cong C_2 \times C_2$. Moreover, if F is a field containing $\mathbb{Q}(i)$, then a plane quartic $X_{a',b',c'}$ is isomorphic to $X_{a,b,c}$ over F if and only if $g(a, b, c) = (a', b', c')$ for some $g \in G$.*

The following result and Corollary 4.1 prove Theorem 0.2.

Corollary 4.5. *Let $X_{a,b,c}$ as before with a^2, b^2, c^2 pairwise distinct. If the field of moduli of $X_{a,b,c}$ relative to the extension \mathbb{C}/\mathbb{R} is \mathbb{R} , then it is a field of definition for $X_{a,b,c}$.*

Proof. By Proposition 4.4, the curve $X_{a,b,c}$ and its conjugate $X_{\bar{a},\bar{b},\bar{c}}$ are isomorphic over \mathbb{C} if and only if $g(a, b, c) = (\bar{a}, \bar{b}, \bar{c})$ for some $g \in G$. It is enough to consider the generators of G .

- i) If $(\bar{a}, \bar{b}, \bar{c}) = g_1(a, b, c) = (b, a, c)$ then $\mu : X_{a,b,c} \rightarrow X_{b,a,c}$, $\mu(x : y : z) = (x : z : y)$ is an isomorphism and $\bar{\mu}\mu = id$.
- ii) If $(\bar{a}, \bar{b}, \bar{c}) = g_2(a, b, c) = (b, c, a)$, i.e., $\bar{a} = b$, $\bar{b} = c$, $\bar{c} = a$, then $a = b = c \in \mathbb{R}$, contradicting the hypothesis on a, b, c . So this case does not appear.
- iii) If $(\bar{a}, \bar{b}, \bar{c}) = g_3(a, b, c) = (-a, -b, c)$ then $\mu : X_{a,b,c} \rightarrow X_{-a,-b,c}$, $\mu(x : y : z) = (ix : y : z)$ is an isomorphism and $\bar{\mu}\mu = id$.
- iv) If $(\bar{a}, \bar{b}, \bar{c}) = g_4(a, b, c) = (a, -b, -c)$ then $\mu : X_{a,b,c} \rightarrow X_{a,-b,-c}$, $\mu(x : y : z) = (x : y : iz)$ is an isomorphism and $\bar{\mu}\mu = id$.

Therefore by Weil's Theorem we conclude that $X_{a,b,c}$ can be defined over \mathbb{R} . \square

We now determine the field of moduli of a plane quartic in the family. Consider the following polynomials invariant for G :

$$j_1(a, b, c) = abc, \quad j_2(a, b, c) = a^2 + b^2 + c^2, \quad j_3(a, b, c) = a^4 + b^4 + c^4;$$

Proposition 4.6. *Let F/K be a general Galois extension with $\mathbb{Q}(i) \subset F \subset \mathbb{C}$ and let $a, b, c \in F$ such that a^2, b^2, c^2 are pairwise distinct and $X_{a,b,c}$ is smooth. The field of moduli of $X_{a,b,c}$ relative to the extension F/K equals $K(j_1, j_2, j_3)$.*

Proof. The morphism $\varphi(a, b, c) = (abc, a^2 + b^2 + c^2, a^4 + b^4 + c^4)$ has degree $24 = |G|$ and clearly $\varphi(g(a, b, c)) = \varphi(a, b, c)$ for any $g \in G$. Thus, by Proposition 4.4, $X_{a,b,c}$ is isomorphic to $X_{a',b',c'}$ over F if and only if $j_k(a, b, c) = j_k(a', b', c')$ for $k = 1, 2, 3$.

Observe that $X_{a,b,c}^\sigma = X_{\sigma(a),\sigma(b),\sigma(c)}$ is isomorphic to $X_{a,b,c}$ over F if and only if for $k = 1, 2, 3$ we have

$$j_k := j_k(a, b, c) = j_k(\sigma(a), \sigma(b), \sigma(c)) = \sigma(j_k(a, b, c)).$$

Thus $U_{F/K}(X_{a,b,c}) = \{\sigma \in \text{Aut}(F/K) : X_{a,b,c}^\sigma \cong X_{a,b,c}\} = \text{Aut}(F/K(j_1, j_2, j_3))$. Since L/K is a general Galois extension we deduce that

$$M_{F/K}(X_{a,b,c}) = \text{Fix}(U_{F/K}(X_{a,b,c})) = K(j_1, j_2, j_3).$$

□

Remark 4.7. Proposition 4.4 can be generalized to the case when F does not contain $\mathbb{Q}(i)$. In this case $X_{a',b',c'}$ is isomorphic to $X_{a,b,c}$ over F if and only if $g(a, b, c) = (a', b', c')$ for some $g \in \langle g_1, g_2 \rangle$ and the field of moduli relative to a general Galois extension F/K equals $K(j_2, j_4, j_5)$ where $j_4(a, b, c) = a+b+c$, $j_5(a, b, c) = a^3+b^3+c^3$.

We now consider the Galois extension $\mathbb{Q}(a, b, c)/\mathbb{Q}(j_1, j_2, j_3)$, assuming that $\mathbb{Q}(i) \subset \mathbb{Q}(a, b, c)$. If σ belongs to the Galois group of such extension, then $X_{a,b,c}^\sigma \cong X_{a,b,c}$ and σ acts on (a, b, c) as some $g_\sigma \in G$ by Proposition 4.4. Thus we can define a natural injective group homomorphism

$$\psi : \text{Aut}(\mathbb{Q}(a, b, c)/\mathbb{Q}(j_1, j_2, j_3)) \rightarrow G, \quad \sigma \mapsto g_\sigma.$$

Observe that, if $a, b, c \in \mathbb{C}$ are generic, then ψ is an isomorphism since the degree of the extension $\mathbb{Q}(a, b, c)/\mathbb{Q}(j_1, j_2, j_3)$ is $24 = |G|$.

Proposition 4.8. *Let $a, b, c \in \mathbb{C}$ such that a^2, b^2, c^2 are pairwise distinct, $X_{a,b,c}$ is smooth and $\mathbb{Q}(i) \subset \mathbb{Q}(a, b, c)$. If $\text{Im}(\psi) \subset \langle g_1, g_2 \rangle$, then $X_{a,b,c}$ can be defined over $\mathbb{Q}(j_1, j_2, j_3) = M_{\mathbb{Q}(a,b,c)/\mathbb{Q}(j_1,j_2,j_3)}(X_{a,b,c})$.*

Proof. According to Weil's Theorem 1.4 we need to choose an isomorphism $f_\sigma : X_{a,b,c} \rightarrow X_{\sigma(a),\sigma(b),\sigma(c)}$ for any $\sigma \in \text{Aut}(\mathbb{Q}(a, b, c)/\mathbb{Q}(j_1, j_2, j_3))$ such that the following condition holds for all σ, τ :

$$(2) \quad f_{\sigma\tau} = f_\tau^\sigma \circ f_\sigma.$$

We assume that $\text{Im}(\psi) = \langle g_1, g_2 \rangle$, the case when there is just an inclusion is similar. Let $\sigma_1 = \psi^{-1}(g_1)$ and $\sigma_2 = \psi^{-1}(g_2)$. We choose $f_{\sigma_1}(x : y : z) = (x : z : y)$, $f_{\sigma_2}(x : y : z) := (z : x : y)$ and $f_\sigma := f_{\sigma_2}^s \circ f_{\sigma_1}^r$ if $\sigma = \sigma_1^r \circ \sigma_2^s$. Observe that f_τ is always defined over \mathbb{Q} , so that $f_\tau^\sigma = f_\tau$. Thus condition (2) clearly holds. □

Example 4.9. Consider a plane quartic $X = X_{a,b,c}$ where $a = \alpha, b = \bar{\alpha}$ with $\alpha \in \mathbb{Q}(i)$ and $c \in \mathbb{Q}$ such that a^2, b^2, c^2 are pairwise distinct and the curve is smooth. By Proposition 4.6 the field of moduli of the curve relative to the extension $\mathbb{Q} \subset \mathbb{Q}(a, b, c) = \mathbb{Q}(i)$ is \mathbb{Q} . The Galois group $\text{Aut}(\mathbb{Q}(i)/\mathbb{Q})$ is generated by the complex conjugation $\sigma(z) = \bar{z}$ and $\psi(\sigma) = g_1$. An isomorphism between X and X^σ is given by $f_\sigma(x : y : z) = (x : z : y)$. Since $\text{id} = f_{\sigma^2} = f_\sigma^\sigma \circ f_\sigma = (f_\sigma)^2$, then X can be defined over \mathbb{Q} .

REFERENCES

1. F. Bars: *Automorphism groups of genus 3 curves*, Number Theory Seminar UAB-UB-UPC on Genus 3 curves. Barcelona, January (2005).
2. E. Bujalance, P. Turbek: *Asymmetric and pseudo-symmetric hyperelliptic surfaces*, Manuscripta Math. 108 (2002), no. 1, 247-256.
3. P. Dèbes, J. C. Douai: *Algebraic covers: field of moduli versus field of definition*, Ann. Sci. École Norm. Sup. (4) 30 No. 3, (1997), 303-338.

4. P. Dèbes, M. Emsalem: *On the field of moduli of curves*, J. Algebra, 211 No. 1 (1999), 42-56.
5. C. J. Earle: *On the moduli of closed Riemann surfaces with symmetries*, Advances in the Theory of Riemann Surfaces. Ann. of Math. Studies 66 (1971), 119-130.
6. R. Hidalgo: *Fields of moduli of regular Fried curves*, Preprint (2010).
7. R. Hidalgo: *Non-hyperelliptic Riemann surfaces with real field of moduli but not definable over the reals*, Arch. Math. 93 (2009), 219-224.
8. E. W. Howe: *Plane quartics with jacobians isomorphic to a hyperelliptic jacobian*, Proceedings of the American Mathematical Society, 129 No. 6 (2000), 1647-1657.
9. B. Huggins: *Fields of moduli of hyperelliptic curves*, Ph.D. Thesis, UCLA, (2005).
10. B. Huggins: *Fields of moduli of hyperelliptic curves*, Math. Res. Lett. 14 (2007), 249-262.
11. S. Koizumi: *Fields of moduli for polarized abelian varieties and for curves*, Nagoya Math. J. 48 (1972), 37-55.
12. A. Kontogeorgis: *Field of moduli versus field of definition for cyclic covers of the projective line*, Jornal de Théorie des Nombres de Bordeaux 21 (2009), 679-692.
13. W. Bosma, J. Cannon, C. Playoust: *The Magma algebra system. I. The user language*, Computational algebra and number theory (London, 1993), J. Symbolic Comput., 24, 1997, 3-4, 235-265.
14. O. Shimura: *On the field of rationality for an abelian variety*, Nagoya Math. J., 45, (1971), 167-178.
15. A. Weil: *The field of definition of a variety*, Amer. J. Math. 78 (1956), 509-524.
16. A. Wootton: *The full automorphism group of a cyclic p -gonal surface*, Journal of Algebra 312 (2007), 377-396.

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